Lecture 9

10. FREQUENCY CHARACTERISTICS OF ELECTRIC CIRCUITS

10.1. Complex Functions of a Circuit

10.1.1. The Concept of the Complex Function of a Circuit

The use of exponential functions allows us to introduce the concept of the complex function of a circuit which is used for describing circuits not containing independent sources of energy.

In the general case, the signal x_{out} at the output of an electric circuit (circuit response) and the input signal x_{in} (stimulus) are related as:

$$a_m \frac{d^m x_{out}}{dt^m} + a_{m-1} \frac{d^{m-1} x_{out}}{dt^{m-1}} + \ldots + a_1 \frac{dx_{bux}}{dt} + a_0 x_{out} = \\ = b_n \frac{d^n x_{in}}{dt^n} + b_{n-1} \frac{d^{n-1} x_{in}}{dt^{n-1}} + \ldots + b_1 \frac{dx_{in}}{dt} + b_0 x_{in}. \quad (10.1)$$

The harmonic functions x_{out}, x_{in} can be represented in the form of an image by complex expressions:

$$\begin{cases} \dot{X}_{in}(t) = \dot{X}_{in} e^{j\omega t}; \\ \dot{X}_{out}(t) = \dot{X}_{out} e^{j\omega t}. \end{cases} \quad (10.2)$$

Replace in (10.1) the real quantities x_{out}, x_{in} with their complex instantaneous quantities. According to (10.2):

$$\left[a_m (j\omega)^m + a_{m-1}(j\omega)^{m-1} + \ldots + a_1 j\omega + a_0\right] \dot{X}_{out} e^{j\omega t} = \\ = \left[b_n (j\omega)^n + b_{n-1}(j\omega)^{n-1} + \ldots + b_1 j\omega + b_0\right] \dot{X}_{in} e^{j\omega t}. \quad (10.3)$$

The complex function of a circuit $K(j\omega)$ is the ratio of the response to a stimulus if they are specified in the form of an exponential function of an imaginary frequency $j\omega$:

$$K(j\omega) = \frac{\dot{X}_{out}(t)}{\dot{X}_{in}(t)} = \frac{\dot{X}_{out} e^{j\omega t}}{\dot{X}_{in} e^{j\omega t}} = \frac{\dot{X}_{out}}{\dot{X}_{in}} = \frac{N(j\omega)}{M(j\omega)} = \\ = \frac{b_n (j\omega)^n + b_{n-1}(j\omega)^{n-1} + \ldots + b_1 j\omega + b_0}{a_m (j\omega)^m + a_{m-1}(j\omega)^{m-1} + \ldots + a_1 j\omega + a_0}. \quad (10.4)$$

From expression (10.3) we get:

$$\dot{X}_{out} = K(j\omega) \dot{X}_{in}.$$

The complex function of a circuit defines its response to the action of an exponential function of an imaginary frequency $j\omega$ in a steady-state mode.

By means of complex functions the theory of analytic functions of a complex variable is linked to the theory of electric circuits.

In circuit analysis, a system of equations is set up in which it is always possible to point out two main quantities that have the meaning of a stimulus and a reaction, such as the current in one branch and the voltage in the other.

Consider the types of complex functions of a circuit. Fig. 10.1 shows a diagram of a passive linear circuit N with the marked input (1–1′) and output (2–2′) terminals. The diagram indicates the input and output currents and voltages as well as the load impedance Z_l connected to the output terminals. There are input and transfer complex functions of a circuit.

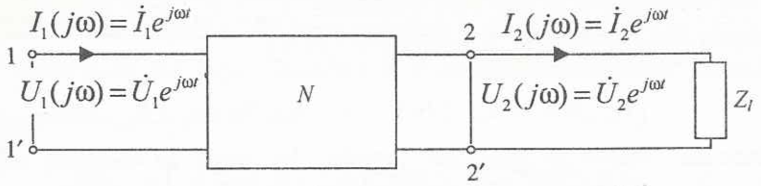


Fig. 10.1

The input complex function of a circuit refers to the ratio of the complex images of the voltage and current acting at the input terminals of the circuit.

They are:

a) the complex input impedance:

$$Z_{in}(j\omega)=\frac{U_1(j\omega)}{I_1(j\omega)}=\frac{\dot{U}_1 e^{j\omega t}}{\dot{I}_1 e^{j\omega t}}=\frac{\dot{U}_1}{\dot{I}_1}; \qquad (10.5)$$

b) the complex input conductance:

$$Y_{in}(j\omega)=\frac{I_1(j\omega)}{U_1(j\omega)}=\frac{\dot{I}_1 e^{j\omega t}}{\dot{U}_1 e^{j\omega t}}=\frac{\dot{I}_1}{\dot{U}_1}. \qquad (10.6)$$

Obviously,

$$Z_{in}(j\omega)=\frac{1}{Y_{in}(j\omega)}.$$

The complex transfer function of a circuit is the ratio of the complex images of the voltage and current acting at different pairs of terminals of the circuit.

They are:

a) the complex voltage transfer ratio:

$$K_U(j\omega)=\frac{U_2(j\omega)}{U_1(j\omega)}=\frac{\dot{U}_2 e^{j\omega t}}{\dot{U}_1 e^{j\omega t}}=\frac{\dot{U}_2}{\dot{U}_1};$$

b) the complex current transfer ratio:

$$K_I(j\omega)=\frac{I_2(j\omega)}{I_1(j\omega)}=\frac{\dot{I}_2 e^{j\omega t}}{\dot{I}_1 e^{j\omega t}}=\frac{\dot{I}_2}{\dot{I}_1};$$

c) the complex transfer impedance:

$$Z_{21}(j\omega)=\frac{U_2(j\omega)}{I_1(j\omega)}=\frac{\dot{U}_2 e^{j\omega t}}{\dot{I}_1 e^{j\omega t}}=\frac{\dot{U}_2}{\dot{I}_1}; \qquad (10.7)$$

d) the complex transfer conductance:

$$Y_{21}(j\omega)=\frac{I_2(j\omega)}{U_1(j\omega)}=\frac{\dot{I}_2 e^{j\omega t}}{\dot{U}_1 e^{j\omega t}}=\frac{\dot{I}_2}{\dot{U}_1}.$$

Obviously,

$$Z_{21}(j\omega)\neq\frac{1}{Y_{21}(j\omega)}. \qquad (10.8)$$

10.1.2. Relationship between Complex Functions and Circuit Parameters

Fig 10.2 shows a diagram of a passive linear circuit N with two marked pairs of terminals: k–k' and l–l'. The voltage source, whose EMF is equal to $\dot{E}_{in}$ and the internal impedance — to Z_{in}, is connected to the terminals k–k'. The load impedance Z_l is connected to the terminals l–l'. Find the relationships between the currents and voltages in this circuit. We will write the equation system according to the loop current method. At the terminals l–l' we will point out the l-th loop with the loop current $\dot{I}_l$, at the terminals k–k' — the k-th loop with the loop current $\dot{I}_k$. Indicate the loop tracing directions. As a result we get:

$$\begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1k} & \dots & Z_{1N} \\ Z_{21} & Z_{22} & \dots & Z_{2k} & \dots & Z_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ Z_{k1} & Z_{k2} & \dots & Z'_{kk} & \dots & Z_{kN} \\ \dots & \dots & \dots & \dots & \dots \\ Z_{N1} & Z_{N2} & \dots & Z_{Nk} & \dots & Z_{NN} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \vdots \\ \dot{I}_k \\ \vdots \\ \dot{I}_N \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dot{E}_{in} \\ \vdots \\ 0 \end{bmatrix}. \qquad (10.9)$$

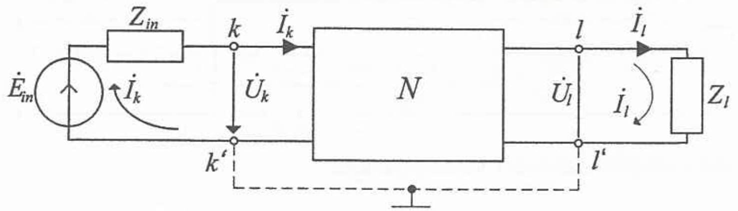


Fig. 10.2

Here N is the total number of independent loops of the circuit (Fig. 10.2). Since the circuit is passive, the loop EMF is present only in the k-th row of the system (10.9), the other loops do not contain any independent energy sources. The impedances Z_{ji}, Z_{jk} are the self and mutual impedances of the loops. The currents $\dot{I}_1, \dot{I}_2, \dots, \dot{I}_k, \dot{I}_N$ are the loop currents. For the k-th loop:

$$Z_{kk} = Z'_{kk} + Z_{in},$$

where Z'_{kk} is the self impedance of the k-th loop with no account of Z_{in}.

That is Z'_{kk} is the impedance of the circuit N from the side of the terminals k–k'.

For the k-th loop we can write according to the loop voltage law:

$$\dot{I}_k Z_{in} + \dot{U}_k - \dot{E}_{in} = 0,$$

hence

$$\dot{I}_k Z_{in} = \dot{E}_{in} - \dot{U}_k. \qquad (10.10)$$

Consider the k-th equation in (10.4):

$$Z_{k1}\dot{I}_1 + Z_{k2}\dot{I}_2 + \dots + Z'_{kk}\dot{I}_k + \dots + Z_{kN}\dot{I}_N = \dot{E}_{in}.$$

Here, taking (10.5) and (10.6) into account:

$$Z_{kk}\dot{I}_k = (Z'_{kk} + Z_{in})\dot{I}_k = Z'_{kk}\dot{I}_k + Z_{in}\dot{I}_k = Z'_{kk}\dot{I}_k + \dot{E}_{in} - \dot{U}_k.$$

Then, we can rewrite:

$$Z_{k1}\dot{I}_1 + Z_{k2}\dot{I}_2 + \dots + Z'_{kk}\dot{I}_k + \dot{E}_{in} - \dot{U}_k \dots + Z_{kN}\dot{I}_N = \dot{E}_{in}$$

or

$$Z_{k1}\dot{I}_1 + Z_{k2}\dot{I}_2 + \dots + Z'_{kk}\dot{I}_k + \dots + Z_{kk}\dot{I}_N = \dot{U}_k.$$

Now the system (10.9) will have the form:

$$\begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1k} & \dots & Z_{1N} \\ Z_{21} & Z_{22} & \dots & Z_{2k} & \dots & Z_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ Z_{k1} & Z_{k2} & \dots & Z'_{kk} & \dots & Z_{kN} \\ \dots & \dots & \dots & \dots & \dots \\ Z_{N1} & Z_{N2} & \dots & Z_{Nk} & \dots & Z_{NN} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \vdots \\ \dot{I}_k \\ \vdots \\ \dot{I}_N \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dot{U}_k \\ \vdots \\ 0 \end{bmatrix}. \qquad (10.11)$$

According to Cramer's rule the solution of (10.11) for the i-th current is

$$\dot{I}_l = \frac{\Delta_l}{\Delta}, \qquad (10.12)$$

where Δ — determinant of the system for the loop impedance matrix; Δ_l — determinant of Δ in which the l-th column is replaced by the column of absolute terms — loop EMF.

$$\Delta_l = \Delta_{kl}\dot{Y}_k = (-1)^{k+l} M_{kl}\dot{U}_k.$$

Here Δ_{ki} is an algebraic adjunct to an element of the k-th row and i-th column; M_{ki} — the minor of the element of the k-th row and l-th column obtained by deleting the k-th row and l-th column from the determinant Δ. Then, according to (10.12):

$$\dot{I}_l = \frac{\Delta_{kl}}{\Delta}\dot{U}_k .$$

Similarly:

$$\dot{I}_k = \frac{\Delta_k}{\Delta} = \frac{\Delta_{kk}}{\Delta}\dot{U}_k . \tag{10.13}$$

The voltage across the load Z_l is

$$\dot{U}_l = \dot{I}_l Z_l = Z_l \frac{\Delta_{kl}}{\Delta}\dot{U}_k . \tag{10.14}$$

From (10.7)–(10.9) we can determine the complex functions of the circuit. From (10.8) we get the complex input impedance:

$$Y_{kk}(j\omega) = \frac{1}{Z_{kk}(j\omega)} = \frac{\dot{I}_k}{\dot{U}_k} = \frac{\Delta_{kk}}{\Delta}. \tag{10.15}$$

From (10.15) we get the complex transfer admittance

$$Y_{lk}(j\omega) = \frac{\dot{I}_l}{\dot{U}_k} = Z_l \frac{\Delta_{kl}}{\Delta}.$$

From (10.13) and (10.14) we get the complex transfer impedance:

$$Z_{lk}(j\omega) = \frac{\dot{U}_l}{\dot{I}_k} = Z_l \frac{\Delta_{kl}}{\Delta_{kk}}.$$

From (10.14) we get the complex voltage transfer ratio:

$$K_{Ulk}(j\omega) = \frac{\dot{U}_l}{\dot{U}_k} = Z_l \frac{\Delta_{kl}}{\Delta}.$$

From (10.23) and (10.13) we get the complex current transfer ratio:

$$K_{Ilk}(j\omega) = \frac{\dot{I}_l}{\dot{I}_k} = Z_l \frac{\Delta_{kl}}{\Delta_{kk}}. \tag{10.16}$$

Thus, the complex functions of a circuit can be expressed by the elements of the loop impedance matrix set up for the passive circuit N and the load impedance Z_l.

A determinant Δ of the N-th order is:

$$\Delta = \sum_{1}^{N!} (-1)^q Z_{1\alpha} Z_{2\beta} \dots Z_{N\nu} .$$

Here q = 1, 2, …, N – the number of rows of the determinant Δ: α, β,…, ν are the permutations from $N!$ of all possible permutations from the numbers 1, 2, …, N. The number q is at the same time the number of inversions in each permutation. It is obvious that the number of terms in (10.16) is also equal to $N!$

Any of the loop impedances in (10.16) $Z_1\alpha$, $Z_2\beta$, …, $Z_{N\nu}$ can be written as a rational function of an imaginary frequency $j\omega$:

$$Z_{q\mu} = r_{q\mu} + j\omega L_{q\mu} + \frac{1}{j\omega C_{q\mu}} = \frac{(j\omega)^2 L_{q\mu} + j\omega r_{q\mu} + C_{q\mu}^{-1}}{j\omega}, \tag{10.17}$$

where μ = 1, 2, …, N.

It is evident that the determinant Δ that consists of the loop resistances $Z_{1\alpha}$, $Z_{2\beta}$, …, $Z_{N\nu}$ is also a rational function of an imaginary frequency $j\omega$. The same can be said about the determinants Δ_{kl}, Δ_{kk}. Therefore, the complex functions of a circuit are rational functions of an imaginary frequency $j\omega$. The coefficients in the circuit complex functions, according to (10.17), are determined only by the circuit parameters r, L, C and, consequently, are real coefficients.

Now we will obtain the circuit complex functions of a circuit (Fig. 10.2) on the basis of the node voltage method. Let the nodes k' and l' be connected, and let their connection point be the basis node. The voltages $\dot{U}_k$, $\dot{U}_l$ will be considered as node voltages. Then the system of node equations for Fig. 10.2 can be written as:

$$\begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1k} & \dots & Y_{1N} \\ Y_{21} & Y_{22} & \dots & Y_{2k} & \dots & Y_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ Y_{k1} & Y_{k2} & \dots & Y_{kk} & \dots & Y_{kN} \\ \dots & \dots & \dots & \dots & \dots \\ Y_{N1} & Y_{N2} & \dots & Y_{Nk} & \dots & Y_{NN} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \vdots \\ \dot{U}_k \\ \vdots \\ \dot{U}_N \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ Y_{in}\dot{E}_{in} \\ \vdots \\ 0 \end{bmatrix}.$$

Here N is the total number of independent nodes. As the circuit is passive, the loop current $Y_{in}\dot{E}_{in}$ is only in the k-th row of the system (10.17), where

$$\dot{Y}_{in} = \frac{1}{Z_{in}}$$

is the internal conductance of the k-th node.

The remaining nodes don't have an energy source. The conductances Y_{jj}, Y_{jk} are intrinsic and transfer conductances of the nodes respectively. The voltages $\dot{U}_1$, $\dot{U}_2$, ..., $\dot{U}_N$ — are node voltages. For the k-th node

$$Y_{kk} = Y'_{kk} + Y_{in},$$

where Y'_{kk} is the self-conductance of the k-th node with no account of Y_{in}. That is, Y'_{kk} is the conductance of the circuit N from the side of the terminals k-k'.

For the k-th node we can be write according to the loop current law:

$$Y_{in}\dot{E}_{in} - Y_{in}\dot{U}_k - \dot{I}_k = 0.$$

From here

$$Y_{in}\dot{U}_k = Y_{in}\dot{E}_{in} - \dot{I}_k.$$

Consider the k-th equation in (10.10):

$$Y_{k1}\dot{U}_1 + Y_{k2}\dot{U}_2 + \dots + Y_{kk}\dot{U}_k + \dots + Y_{kN}\dot{U}_N = Y_{in}\dot{E}_{in} \quad (10.18)$$

where, taking into account expressions (10.18) and (10.19):

$$Y_{kk}\dot{U}_k = (Y'_{kk} + Y_{in})\dot{U}_k = Y'_{kk}\dot{U}_k + Y_{in}\dot{U}_k = Y'_{kk}\dot{U}_k + Y_{in}\dot{E}_{in} - \dot{I}_k.$$

Equation (10.18) can be rewritten as:

$$Y_{k1}\dot{U}_1 + Y_{k2}\dot{U}_2 + \dots + Y'_{kk}\dot{U}_k + Y_{in}\dot{E}_{in} - \dot{I}_k + \dots + Y_{kN}\dot{U}_N = Y_{in}\dot{E}_{in},$$

or

$$Y_{k1}\dot{U}_1 + Y_{k2}\dot{U}_2 + \dots + Y'_{kk}\dot{U}_k + \dots + Y_{kN}\dot{U}_N = \dot{I}_k.$$

Now the system (10.17) will have the form:

$$\begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1k} & \dots & Y_{1N} \\ Y_{21} & Y_{22} & \dots & Y_{2k} & \dots & Y_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ Y_{k1} & Y_{k2} & \dots & Y'_{kk} & \dots & Y_{kN} \\ \dots & \dots & \dots & \dots & \dots \\ Y_{N1} & Y_{N2} & \dots & Y_{Nk} & \dots & Y_{NN} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \vdots \\ \dot{U}_k \\ \vdots \\ \dot{U}_N \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dot{I}_k \\ \vdots \\ 0 \end{bmatrix}. \quad (10.19)$$

The solution of the system (10.19) for the l-th current:

$$\dot{U}_l = \frac{\Delta_l}{\Delta}. \quad (10.20)$$

Expanding the determinant Δ_l into the elements of the l-th column, we get:

$$\Delta_l = \Delta_{kl}\dot{I}_k = (-1)^{k+1}M_{kl}\dot{I}_k.$$

From (10.20):

$$\dot{U}_l = \frac{\Delta_{kl}}{\Delta}\dot{I}_k. \quad (10.21)$$

Similarly:

$$\dot{U}_k = \frac{\Delta_k}{\Delta} = \frac{\Delta_{kk}}{\Delta}\dot{I}_k. \quad (10.22)$$

The current in the load Z_l:

$$\dot{I}_l = \dot{U}_l Y_l = Y_l\frac{\Delta_{kl}}{\Delta}\dot{I}_k, \quad (10.23)$$

where $Y_l = \dfrac{1}{Z_l}$ — conductance of the load.

$$Z_{kk}(j\omega)=\frac{1}{Y_{kk}(j\omega)}=\frac{\dot{U}_k}{\dot{I}_k}=\frac{\Delta_{kk}}{\Delta}.$$

From (10.21) determine the complex transfer impedance:

$$Z_{lk}(j\omega)=\frac{\dot{U}_l}{\dot{I}_k}=\frac{\Delta_{kl}}{\Delta}.$$

Determine the complex transfer conductance:

$$Y_{lk}(j\omega)=\frac{\dot{I}_l}{\dot{U}_k}=Y_l\frac{\Delta_{kl}}{\Delta_{kk}}.$$

From (10.23) determine the complex current transfer ratio:

$$K_{Ilk}(j\omega)=\frac{\dot{I}_l}{\dot{I}_k}=Y_l\frac{\Delta_{kl}}{\Delta}.$$

From (10.21)–(10.22) determine the complex voltage transfer ratio:

$$K_{Ulk}(j\omega)=\frac{\dot{U}_l}{\dot{U}_k}=\frac{\Delta_{kl}}{\Delta_{kk}}.$$

Thus, the complex functions of a circuit can be expressed by the elements of the ~~loop impedance~~ matrix set up for the passive circuit N and the load impedance Y_l. node voltages

A determinant of the N-th order Δ is:

$$\Delta=\sum_{1}^{N!}(-1)^q\,Y_{1\alpha}Y_{2\beta}\ldots Y_{N\nu}.$$

The node conductance is:

$$Y_{q\mu}=g_{q\mu}+j\omega C_{q\mu}+\frac{1}{j\omega L_{q\mu}}=\frac{(j\omega)^2C_{q\mu}+j\omega g_{q\mu}+L_{q\mu}^{-1}}{j\omega}.$$

Comparing the expressions for the complex functions of a circuit obtained by the loop current method (LCM) and the expressions obtained by the node voltage method (MNV), we find that

$$Y_{kk}(j\omega)=\left(\frac{\Delta_{kl}}{\Delta}\right)_{MLC}=\left(\frac{\Delta}{\Delta_{kk}}\right)_{MNV};$$

$$Y_{lk}(j\omega)=\left(\frac{\Delta_{kl}}{\Delta}\right)_{MLC}\neq\left(\frac{\Delta}{\Delta_{kl}}\right)_{MNV}=\frac{1}{Z_{lk}(j\omega)};$$

$$Z_{lk}(j\omega)=\left(Z_l\frac{\Delta_{kl}}{\Delta_{kk}}\right)_{MLC}\neq\left(\frac{1}{Y_l}\frac{\Delta_{kk}}{\Delta_{kl}}\right)_{MNV}=\frac{1}{Y_{lk}(j\omega)}.$$

10.2. Complex Function Components and Frequency Characteristics of Electric Circuits

A complex function, as any complex number, can be represented in algebraic, exponential, and trigonometric forms:

$$K(j\omega)=\dot{K}(\omega)=\frac{\dot{X}_{out}}{\dot{X}_{in}}=R(\omega)+jX(\omega)=K(\omega)e^{j\varphi(\omega)}=$$
$$=K(\omega)[\cos\varphi(\omega)+j\sin\varphi(\omega)],$$

where $R(\omega)=\mathrm{Re}[K(j\omega)]=K(\omega)\cos\varphi(\omega)$ — real part of a CFC;

$X(\omega)=\mathrm{Im}[K(j\omega)]=K(\omega)\sin\varphi(\omega)$ — imaginary part of a CFC;

$\varphi(\omega)=\mathrm{Arg}[K(j\omega)]=\mathrm{atan}\dfrac{X(\omega)}{R(\omega)}$ — argument of a CFC;

$K(\omega)=Mod[K(j\omega)]=\sqrt{[R(\omega)]^2+[X(\omega)]^2}$ — module of a CFC.

The quantities $R(\omega)$, $X(\omega)$, $\varphi(\omega)$, $K(\omega)$ depend on the frequency and are called real (RFC), imaginary (IFC), phase (PhFC) and amplitude (AFC) frequency characteristics respectively. They are built by plotting the frequency ω on the abscissa and the frequency characteristic — on the ordinate.

For example, for a series r, L, C circuit a typical AFC and PhFC have the forms as in Fig. 10.3, *a*, *b*.

The amplitude-frequency (AFC) and phase-frequency (PhFC) characteristics can be united into one amplitude-phase-frequency characteristic (APhFC). It is plotted in relation to the coordinates $X(\omega)$, $R(\omega)$ when the frequency ω varies from 0 to ∞ or, in the general case, from $-\infty$ to ∞. For each value of frequency there is a value $R(\omega)$ which is laid off as abscissa and $X(\omega)$ is laid off as ordinate. The values $X(\omega)$ and $R(\omega)$

define the point on the plane $X(\omega)$, $R(\omega)$.The geometric location of the points which are obtained with different values of ω make an APhFC curve.

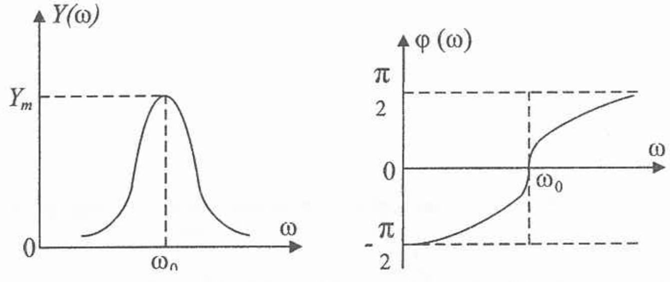


Fig. 10.3

Fig 10.4 shows the APhFC plot for a second-order circuit. Here at $\omega = 0$ the value $X(0) = 0$. Since $R(0) \neq 0$, the point of the characteristic is on the abscissa. When the frequency increases, values of $X(\omega)$ appear and the values of $R(\omega)$ decrease. So, at $\omega = \omega_1$ we get $X(\omega_1) < 0$, $R(\omega_1) < R(0)$. At $\omega_2 > \omega_1$, we get the next point of the characteristic in the same way, and so on. At $\omega \to \infty$, $X(\infty) \to 0$, $R(\infty) \to 0$ and the characteristic's point coincides with the beginning of the coordinates.

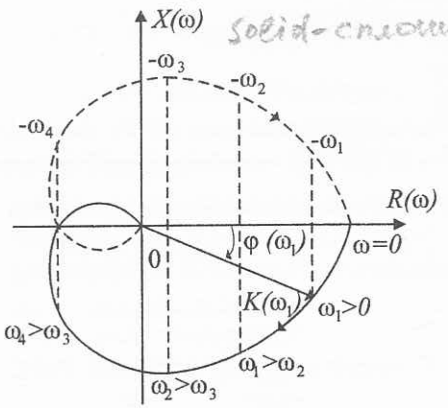


Fig 10.4

Quite often an APhFC is supplemented with a curve at $\omega < 0$. In this case in Fig. 10.4 a dotted curve appears which is symmetrical to the characteristic at $\omega > 0$. The APhFC can also be built by calculating the modulus $K(\omega)$ and the argument $\varphi(\omega)$ for each value of the frequency ω. Laying off the segment $K(\omega)$ in the direction of $\varphi(\omega)$, we get the characteristic point.

At the end of the segment an arrow is placed and we say that with varying frequency the vector $K(\omega)$ describes a curve which is called the frequency locus. The arrow on the curve indicates the direction in which the frequency ω increases.

Sometimes the frequency locus is also called the Nyquist diagram. In the theory of electric filters the exponential notation is used for the complex functions of a circuit:

$$K(j\omega) = e^{\gamma(j\omega)} = e^{[\alpha(\omega) + j\beta(\omega)]}. \qquad (10.24)$$

Here $\gamma(j\omega)$ is the complex propagation coefficient.

Takings the logarithm of (10.24), we get:

$$\ln K(j\omega) = \ln[K(\omega)e^{\gamma(j\omega)}] = \ln K(\omega) + j\varphi(\omega) = \alpha(\omega) + j\beta(\omega);$$

$$\alpha(\omega) = \ln K(\omega); \beta(\omega) = \varphi(\omega). \qquad (10.25)$$

The value $\alpha(\omega)$ is called the logarithmic amplitude-frequency characteristic (LAFC). It is measured in Napiers (Np) if calculated in terms of the natural logarithm according to (10.25). If $\alpha(\omega)$ is calculated in terms of the decimal logarithm, then the measurement unit is called the Bell (B):

$$\alpha(\omega) = 2\lg K(\omega).$$

Since

$$2\ln K(\omega) = \frac{2}{\ln 10}\ln K(\omega) = 0{,}86861 \cdot \ln K(\omega),$$

then

$$1\ \text{Np} = 0{,}8686\ \text{B}, \quad 1\ \text{B} = 1{,}15\ \text{Np}, \quad 1\ \text{dB} = 0{,}1\ \text{B}.$$

10.3. Frequency Characteristics of Simplest Circuits

10.3.1 Frequency Characteristics of First-Order Circuits

Consider a simplest rC – circuit (Fig. 10.5, *a*). Let us determine the complex voltage transfer ratio

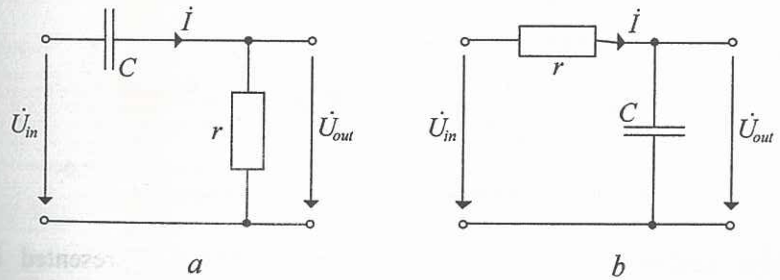


Fig. 10.5

$$K_U(j\omega) = K(\omega)\,e^{j\varphi(\omega)} = \frac{\dot U_{out}}{\dot U_{in}} = \frac{\dot I r}{\dot I Z} = \frac{r}{r + \frac{1}{j\omega C}} =$$

$$= \frac{j\omega rC}{1 + j\omega rC} = \frac{(\omega rC)^2}{1 + (\omega rC)^2} + j\frac{\omega rC}{1 + (\omega rC)^2} = \qquad (10.26)$$

$$= \frac{(\omega\tau)^2}{1 + (\omega\tau)^2} + j\frac{\omega\tau}{1 + (\omega\tau)^2} = R(\omega) + jX(\omega).$$

Here $\tau = rC$ — time constant of the circuit.
Then

$$R(\omega) = \frac{(\omega\tau)^2}{1 + (\omega\tau)^2}$$ — real frequency characteristic;

$$X(\omega) = \frac{\omega\tau}{1 + (\omega\tau)^2}$$ — imaginary frequency characteristic;

$$K(\omega) = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}}$$ — amplitude frequency characteristic;

$$\varphi(\omega) = \operatorname{atan}\frac{1}{\omega\tau} = \frac{\pi}{2} - \operatorname{atan}(\omega\tau)$$ — phase-frequency characteristic.

The diagrams of RFC, IFC, AFC, PhFC, and APhFC are presented in Fig. 10.6, *a*, *b*, *c*, *d*, *e* respectively.

Let us consider the rC — circuit in Fig.10.5, *b*. For this circuit:

$$K_U(j\omega) = \frac{\frac{1}{j\omega C}}{r + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega rC} = \frac{1}{1 + (\omega rC)^2} - j\frac{\omega rC}{1 + (\omega rC)^2} =$$

$$= R(\omega) + jX(\omega).$$

Here RFC, IFC, AFC, PhFC are:

$$R(\omega) = \frac{1}{1 + (\omega\tau)^2}; \qquad X(\omega) = -\frac{\omega\tau}{1 + (\omega\tau)^2};$$

$$K(\omega) = \frac{1}{\sqrt{1 + (\omega\tau)^2}}; \qquad \varphi(\omega) = -\operatorname{artan}(\omega\tau).$$

The diagrams of the frequency characteristics are presented in Fig.10.7, *a*, *b*, *c*, *d*, *e* respectively.

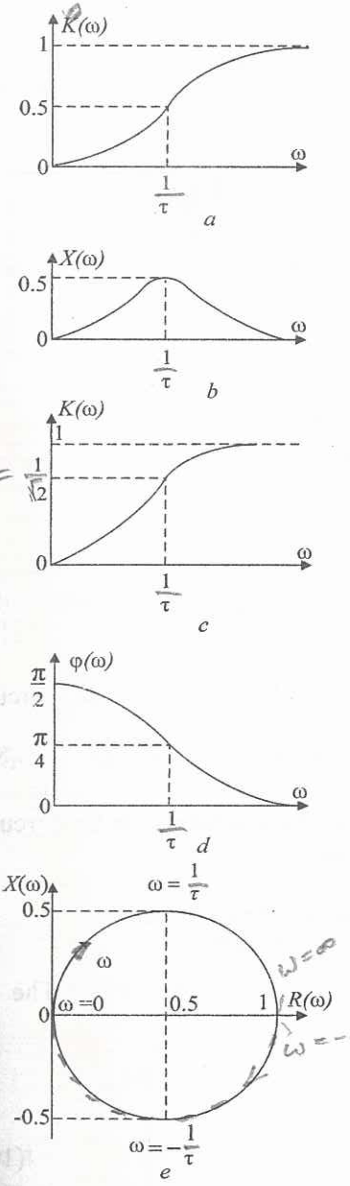


Fig. 10.6

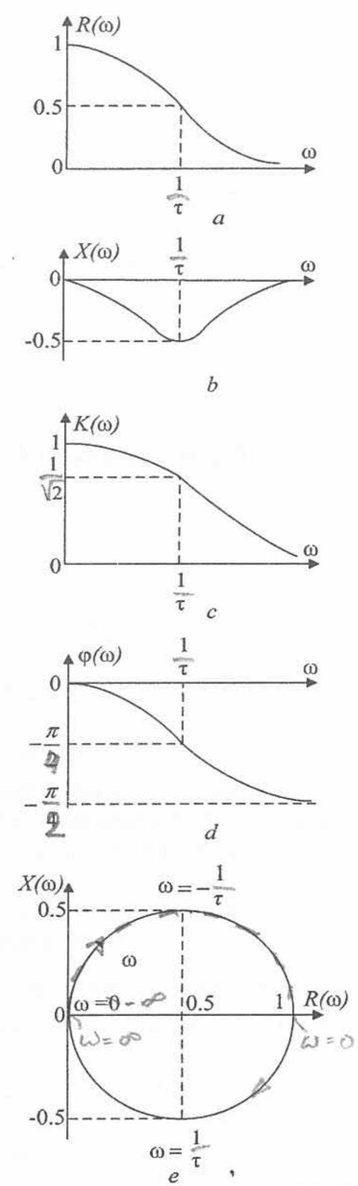
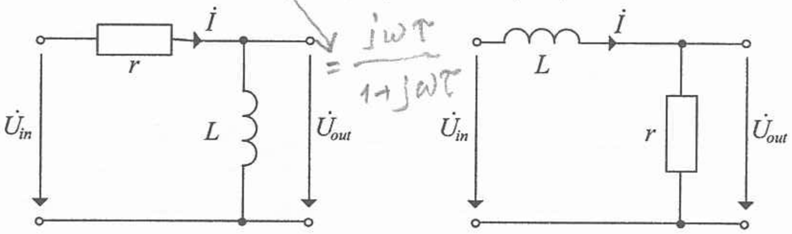


Fig. 10.7

Consider the rL — circuit in Fig. 10.8, a. For this circuit:

$$K_U(j\omega)=\frac{j\omega L}{1+j\omega L}=\frac{(\omega L)^2}{r^2+(\omega L)^2}+j\frac{\omega rL}{r^2+(\omega L)^2}=R(\omega)+jx(\omega).$$



a *b*

Fig. 10.8

Denote the time constant of the circuit by τ:

$$\frac{L}{r}=\tau,$$

then

$$K_U(j\omega)=\frac{(\omega\tau)^2}{1+(\omega\tau)^2}+j\frac{\omega\tau}{1+(\omega\tau)^2}. \qquad (10.27)$$

Expression (10.27) coincides with (10.26) for the RC-circuit in Fig. 10.5, a.

Consequently, the frequency characteristics of the circuits in Fig. 10.8, a and 10.5, a are identical.

Analysis shows that the frequency characteristics on the circuits in Fig. 10.8, b and 10.5, b are also identical.

10.3.2. Frequency Characteristics of Second- and Third-Order Circuits

Consider the second-order circuit shown in Fig. 10.9. The node conductance matrix for this circuit is:

$$\begin{bmatrix} j\omega C & -j\omega C & 0; \\ -j\omega C & j\omega C+\dfrac{2}{r} & -\dfrac{1}{r}; \\ 0 & -\dfrac{1}{r} & j\omega C+\dfrac{1}{r}. \end{bmatrix} \qquad (10.28)$$

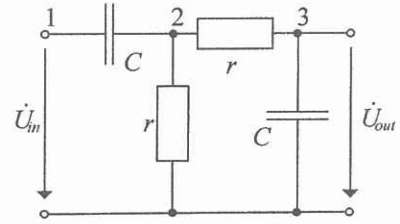


Fig. 10.9

Hence, the algebraic adjuncts Δ_{13}, Δ_{11} are:

$$\Delta_{13}=(-1)^{1+3}\mathbf{M}_{13};\ \Delta_{11}=(-1)^{1+1}\mathbf{M}_{11}.$$

Here $\mathbf{M}_{13}$ and $\mathbf{M}_{11}$ are the minor-determinants obtained from the matrix determinant (10.28) by deleting the first row and the third column and the first row and the first column respectively, that is

$$\Delta_{13}=-(j\omega C)\left(-\frac{1}{r}\right)=\frac{j\omega C}{r};$$

$$\Delta_{11}=\left(j\omega C+\frac{2}{r}\right)\left(j\omega C+\frac{1}{r}\right)-\left(-\frac{1}{r}\right)\left(-\frac{1}{r}\right)=$$

$$=\frac{-\omega^2r^2C^2+3j\omega rC+1}{r^2}.$$

The voltage transfer ratio is:

$$K_{U31}(j\omega)=K_{U31}(\omega)\,e^{j\varphi_{31}(\omega)}=\frac{\Delta_{31}}{\Delta_{11}}=\frac{j\omega rC}{-\omega^2r^2C^2+3j\omega rC+1}.$$

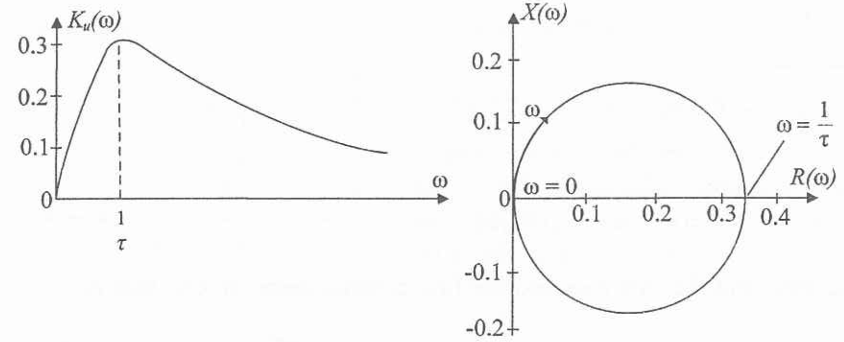
Here

$$K_{U31}(\omega)=\frac{\omega rC}{\sqrt{\left(1-\omega^2r^2C^2\right)^2+9\omega^2r^2C^2}}=$$
$$=\frac{\omega\tau}{\sqrt{\left(1-\omega^2\tau^2\right)^2+9\omega^2\tau^2}}; \qquad (10.29)$$

$$\varphi_{31}(\omega)=\frac{\pi}{2}-\operatorname{atan}\frac{3\omega rC}{1-\omega^2r^2C^2}=\frac{\pi}{2}-\operatorname{atan}\frac{3\omega\tau}{1-\omega^2\tau^2}, \qquad (10.30)$$

where $\tau=rC$ is the time constant.

The diagrams of AFC and APhFC are given in Fig. 10.10, a, b respectively.



a *b*

Fig. 10.10

Consider the second-order circuit shown in Fig. 10.11 (a Wien bridge). The node conductance matrix for this circuit is:

$$\begin{bmatrix} \dfrac{1}{r+\dfrac{1}{j\omega C}} & -\dfrac{1}{r+\dfrac{1}{j\omega C}} \\ -\dfrac{1}{r+\dfrac{1}{j\omega C}} & \dfrac{1}{r+\dfrac{1}{j\omega C}}+\dfrac{1}{r}+j\omega C \end{bmatrix}.$$

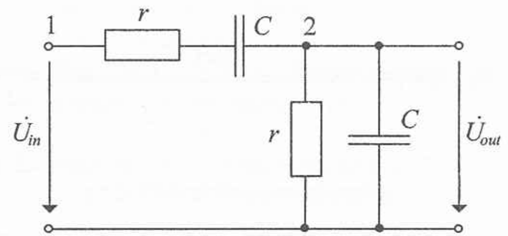


Fig. 10.11

Hence, the algebraic adjuncts Δ_{12}, Δ_{11} are

$$\Delta_{12}=(-1)^{1+2}\left(-\frac{1}{r+\dfrac{1}{j\omega rC}}\right)=\frac{j\omega C}{1+j\omega rC};$$

$$\Delta_{11}=(-1)^{1+1}\left(-\frac{1}{r+\dfrac{1}{j\omega C}}+\frac{1}{r}+j\omega C\right)=\frac{1-\omega^2r^2C^2+3jvrC}{r(1+j\omega rC)}.$$

The voltage transfer ratio will be:

$$K_{U21}(j\omega)=K_{U21}(\omega)\,e^{j\varphi_{21}(\omega)}=\frac{\Delta_{21}}{\Delta_{11}}=$$

$$=\frac{j\omega rC\,(1+j\omega rC)}{(1+j\omega rC)\left(1-\omega^2r^2C^2+3j\omega rC\right)}=$$

$$=\frac{j\omega rC}{1-\omega^2r^2C^2+3j\omega rC}.$$

Here

$$K_{U21}(\omega)=\frac{\omega rC}{\sqrt{\left(1-\omega^2r^2C^2\right)^2+9\omega^2r^2C^2}}=\frac{\omega\tau}{\sqrt{\left(1-\omega^2\tau^2\right)^2+9\omega^2\tau^2}} \qquad (10.31)$$

$$\varphi_{21}(\omega)=\frac{\pi}{2}-\text{atan}\,\frac{3\omega rC}{1-\omega^2r^2C^2}=\frac{\pi}{2}-\text{atan}\,\frac{3\omega\tau}{1-\omega^2\tau^2}. \qquad (10.32)$$

When comparing (10.31) and (10.32) with (10.29) and (10.30), we can see that these expressions are identical and, consequently, the frequency characteristics of the circuits in Fig.10.9 and 10.11 coincide.

Consider the third-order circuit shown in Fig. 10.12 (a double T-bridge).

The node conductance matrix for this circuit is:

$$\begin{bmatrix} \dfrac{1}{r}+j\omega C & -j\omega C & -\dfrac{1}{r} & 0 \\ -j\omega C & 2j\omega C+\dfrac{2}{r} & 0 & -j\omega C \\ -\dfrac{1}{r} & 0 & 2j\omega C+\dfrac{2}{r} & -\dfrac{1}{r} \\ 0 & -j\omega C & -\dfrac{1}{r} & \dfrac{1}{r}+j\omega C \end{bmatrix}.$$

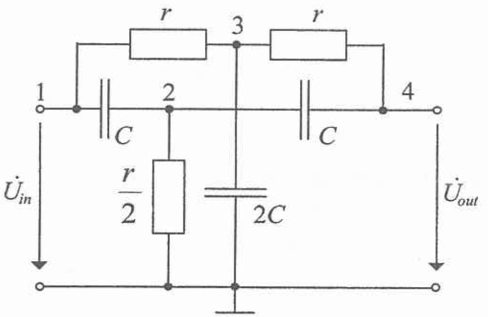


Fig. 10.12

Hence, the algebraic adjuncts Δ_{14}, Δ_{11} are:

$$\Delta_{14}=(-1)^{1+4}\begin{bmatrix} -j\omega C & 2j\omega C+\frac{2}{r} & 0\\ -\frac{1}{r} & 0 & 2j\omega C+\frac{2}{r}\\ 0 & -j\omega C & -\frac{1}{r}\end{bmatrix}=$$

$$=-\left[-\left(-\frac{1}{r}\right)\left(2j\omega C+\frac{2}{r}\right)\left(-\frac{1}{r}\right)-(-j\omega C)\left(2j\omega C+\frac{2}{r}\right)(-j\omega C)\right]=$$

$$=\frac{2(1-\omega^2r^2C^2)(1+j\omega rC)}{r^3};$$

$$\Delta_{11}=(-1)^{1+1}\begin{bmatrix} 2j\omega C+\frac{2}{r} & 0 & -j\omega C\\ 0 & 2j\omega C+\frac{2}{r} & -\frac{1}{r}\\ -j\omega C & -\frac{1}{r} & j\omega C+\frac{1}{r}\end{bmatrix}=$$

$$=\frac{2(1+j\omega rC)(1-\omega^2r^2C^2+4j\omega rC)}{r^3}.$$

The voltage transfer ratio is:

$$K_{U41}(j\omega)=K_{U41}(\omega)\,e^{j\varphi_{41}(\omega)}=\frac{\Delta_{41}}{\Delta_{11}}=$$

$$=\frac{2(1-\omega^2r^2C^2)(1+j\omega rC)r^3}{2(1+j\omega rC)(1-\omega^2r^2C^2+4j\omega rC)r^3}=$$

$$=\frac{1-\omega^2r^2C^2}{1-\omega^2r^2C^2+4j\omega rC}.$$

Here

$$K_{U41}(\omega)=\frac{\left|1-\omega^2\tau^2\right|}{\sqrt{(1-\omega^2\tau^2)^2+16\omega^2\tau^2}};$$

$$\varphi_{41}(\omega)=-\text{atan}\frac{4\omega rC}{1-\omega^2r^2C^2}=-\text{atan}\frac{4\omega\tau}{1-\omega^2\tau^2}.$$

The diagrams of the AFC and APHFC are presented in Fig. 10.13, *a*, *b*.

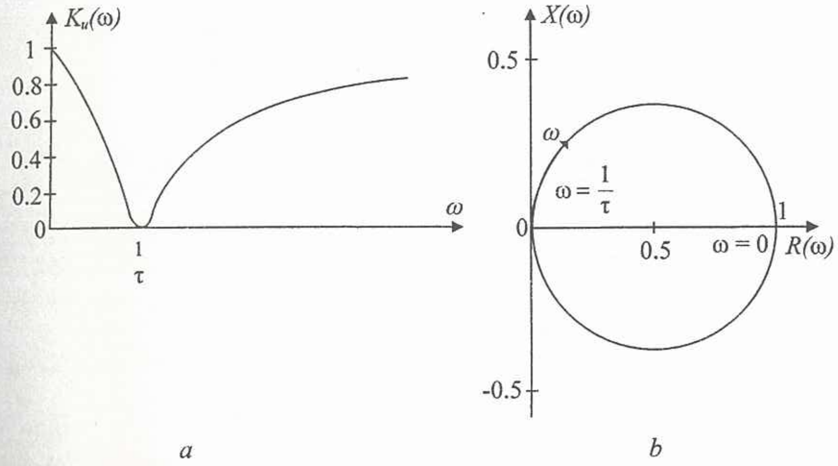


Fig. 10.13

Example 1

For the circuit in Fig. 10.14, *a*, find the complex transfer function $K_u(j\omega)=\frac{U_2}{U_1}$ (complex voltage transfer ratio).

Build qualitative graphs of the AFC and PhFC.

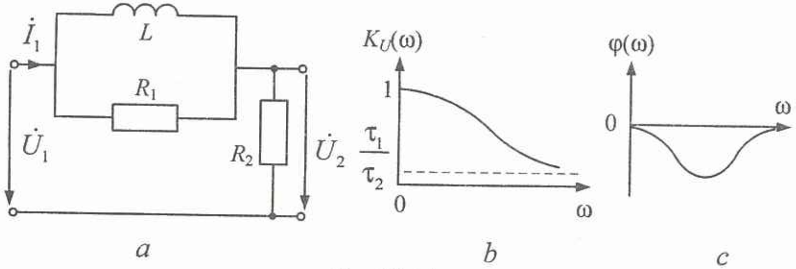


Fig. 10.14

Solution

The complex voltage transfer ratio is:

$$K(j\omega)=\frac{\dot{U}_2}{\dot{U}_1}=\frac{I_1R_2}{I_2(R_2+Z_{LR})}=\frac{R_2}{R_2+Z_{LR}};\ Z_{LR}=\frac{R_1 j\omega L}{R_1+j\omega L}.$$

Then

$$K_u(j\omega)=\frac{R_2}{R_2+\dfrac{R_1 j\omega L}{R_1+j\omega L}}=\frac{R_2(R_1+j\omega L)}{R_1R_2+R_2 j\omega L+R_1 j\omega L}$$

$$=\frac{R_1R_2+R_2 j\omega L}{R_1R_2+(R_1+R_2)j\omega L}=\frac{1+j\omega\dfrac{L}{R_1}}{1+j\omega L\dfrac{R_1+R_2}{R_1R_2}}=\frac{1+j\omega\tau_1}{1+j\omega\tau_2}=$$

$$=\frac{\sqrt{1+(\omega\tau_1)^2}\,e^{j\mathrm{atan}\omega\tau_1}}{\sqrt{1+(\omega\tau_2)^2}\,e^{j\mathrm{atan}\omega\tau_2}}=\frac{\sqrt{1+(\omega\tau_1)^2}}{\sqrt{1+(\omega\tau_2)^2}}e^{j\mathrm{atan}\omega\tau_1-j\mathrm{atan}\omega\tau_2}.$$

Here: $\tau_1=\dfrac{L}{R_1}$; $\tau_2=\dfrac{L(R_1+R_2)}{R_1R_2}$ — time constants.

Hence, the AFC and PhFC are:

$$K_u(\omega)=\frac{\sqrt{1+(\omega\tau_1)^2}}{\sqrt{1+(\omega\tau_2)^2}};\ \varphi(\omega)=\mathrm{atan}\omega\tau_1-\mathrm{atan}\omega\tau_2.$$

The graphs of $K_u(\omega)$ — AFC and of $\varphi(\omega)$ — PhFC are shown in Fig. 10.1, *b*, *c*.